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Quantum black holes: the event horizon as a fuzzy sphere

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Abstract: Modeling the event horizon of a black hole by a fuzzy sphere leads us to modify some suggestions in the literature concerning black hole mass spectra. We derive a formula for the mass spectrum of quantum black holes in terms of four integers which define the area, angular momentum, electric and magnetic charge of the black hole. Although the event horizon becomes a commutative sphere in the classical limit a vestige of the quantum theory still persists in that the event horizon stereographically projects onto the non-commutative plane. We also suggest how the classical bounds on extremal black holes might be modified in the quantum theory.

Keywords: Models of Quantum Gravity, Non-Commutative Geometry.

Contents

1. Introduction

Bekenstein's suggestion that the surface area of a black hole should be quantised in multiples of the Planck area, and that the entropy should be proportional to the area [1], was triumphantly vindicated by Hawking's calculation of the black hole temperature and entropy as a function of area [2]. Quantising the event horizon is very reminiscent of the concept of a "fuzzy sphere", S^2_F [3], in which points are "smeared out" and the geometry becomes non-local. In this paper we shall investigate modeling a black hole event horizon with a fuzzy sphere and show that this idea fits nicely with many of Bekenstein's suggestions of treating a black hole as a particle, [4, 5] (a point of view also strongly advocated by 't Hooft [6]).

Bekenstein suggested that the area of a black hole should have a quantised spectrum

$$A = a(N+\eta)l_P^2\,, \tag{1.1}$$

with $N = 1, 2\ldots$, and $a > 0$, $\eta > -1$ undetermined constants ($l_P = \sqrt{G_N\hbar/c^3}$ is the Planck length). This idea has since been developed further in [7] and discretisation of the horizon has also been postulated by 't Hooft [8]. It was suggested some time ago that a black hole event horizon might be modeled by a fuzzy sphere [9].

Part of the characterisation of a fuzzy sphere is an irreducible representation of SU(2) of dimension $N = 2k+1$, with k either integral or half-integral. Functions on the fuzzy sphere are then represented by $N \times N$ matrices acting on an N-dimensional Hilbert space. We argue in the following that it is natural to take the area of the event horizon to be

$$A = 4\pi(2k+1)l_P^2 \tag{1.2}$$

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so that $a = 4\pi$ and $\eta = 0$ above. The value of $a = 4\pi$ that is natural in a fuzzy sphere construction has also been found in the semi-classical approach of [10] and a mini-superspace approach to black hole quantisation in [11, 12]. An equal spaced area spectrum like that of (1.2) was found in [13] though the prefactor was undetermined,

In addition it will emerge from our analysis that, in the classical limit $N \to \infty$, the neighbourhood of a point on the event horizon can be stereographically projected onto a non-commutative plane with non-commutativity parameter

$$\theta = l_P^2 \,. \tag{1.3}$$

A relation between quantisation of the event horizon area and the non-commutative plane was suggested in [14]. Non-commutativity on the event horizon was also suggested in [15] and a direct approach to deriving non-commutativity in black hole physics was recently initiated in [16].

First we briefly summarise the results of our analysis. With the values $a = 4\pi$ and $\eta = 0$ above we show that the mass spectrum for black holes suggested by Bekenstein [5] is modified to give:

$$M^2_{k,j,q_e} = \left\{ \frac{\left(2k+1+\alpha q_e^2\right)^2 + 4j(j+1)}{4(2k+1)} \right\} {m_P}^2 \,, \tag{1.4}$$

where j is integral or half-integral and q_e is an integer, representing angular momentum and electric charge respectively, $\alpha = e^2/\hbar c$ is the fine structure constant and $m_P = \sqrt{c\hbar/G_N}$ is the Planck mass (there is a modification of this formula when magnetic monopoles are included). The smallest possible mass for a black hole in this scheme is therefore

$$M = \frac{1}{2} m_P \,, \tag{1.5}$$

when $k = j = q_e = 0$.

For given j and q_e the quantum number k is bounded below by

$$(2k+1)^2 \geq 4j(j+1) + \alpha^2 q_e^4 \,. \tag{1.6}$$

In particular, for a zero charge black hole, the classical bound

$$J^2 \leq M^4 \tag{1.7}$$

(in units with $G_N = c = 1$) is replaced by

$$J^2 \leq M^4 - \frac{\pi^2 l_P^4}{A^2} \hbar^2 \,. \tag{1.8}$$

The layout of the paper is as follows. In section 2 the quantisation of the area arising from the fuzzy sphere hypothesis is discussed for Schwarzschild black holes and the projection to the non-commutative plane is explained. Section 3 analyses non-zero angular momentum and the associated bounds on the mass while section 4 does the same for charged and rotating holes. The relation to entropy is discussed in section 5 and the results are summarised in section 6.

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2. Schwarzschild black holes

The 2-dimensional sphere is a symplectic manifold — a phase-space in physics language, albeit a compact one. This phase-space can be quantised to give S_F^2. The concept of a point on S_F^2 is not defined but instead the points are smeared out into a finite number of phase-space 'cells', hence the name 'fuzzy', [3]. For any integer, $N = 2k+1$ with k labelling SU(2) representations either integral or half-integral, S_F^2 has N cells and operators on phase-space are $N \times N$ matrices acting on a N-dimensional Hilbert space, [17]. Visually S_F^2 might be viewed as being like the surface of Jupiter, with the belts being unit cells, but this is not essential since, as in any quantum phase space, only the area of the fundamental cells, not their shape, is fixed.

If we picture the event horizon of a black hole as a fuzzy sphere then the total area of the event horizon is naturally a multiple of the area of a fundamental unit cell. Suppose the unit cells have area al_P^2, with a a positive dimensionless constant of order one. Then the total area of the event horizon is

$$A = Nal_P^2\,, \tag{2.1}$$

and, since $N = 2k + 1$, we conclude that $\eta = 0$ in equation (1.1).

For a non-rotating black hole with zero charge (2.1) immediately implies that the Schwarzschild radius R_S is also quantised

$$R_S^2 = \frac{A}{4\pi} = \frac{Nal_P^2}{4\pi}\,. \tag{2.2}$$

To avoid messy factors of 4π it is convenient to define $\bar{A} = A/4\pi$ and $\bar{a} = a/4\pi$ so

$$R_S^2 = \bar{A} = N\bar{a}l_P^2\,. \tag{2.3}$$

The mass of the hole can now be expressed as

$$M = \frac{R_S c^2}{2G_N} = \sqrt{N\bar{a}}\frac{l_P c^2}{2G_N} = \sqrt{N\bar{a}}\frac{m_P}{2}\,. \tag{2.4}$$

The hypothesis that the event horizon is a fuzzy-sphere thus immediately leads us to conclude that black hole masses are quantised

$$M^2 = \frac{N\bar{a}}{4}{m_P}^2 \tag{2.5}$$

with N a positive integer, a formula that was postulated in [1].

For astrophysical black holes N is so large that the quantum nature of the mass would be unobservable, but in the final stages of black hole evaporation the black hole would go through a series of discrete states until the final state is reached, with $N = 1$ (i.e. $k = 0$) and residual mass $M_0 = \sqrt{\bar{a}}m_P/2$. Thus in this picture evaporating black holes do not disappear but must necessarily leave behind a residual hole of the order of the Planck mass. As remarked in [4] the situation is reminiscent of the Bohr model of the atom in which orbiting electrons can only occupy a discrete set of orbits, dictated by the Bohr-Sommerfeld

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constraint $\oint pdq = 2\pi N\hbar$ on the orbitals, and decaying electrons must finally lodge in the ground state thus rendering atoms stable.

Non-commuting co-ordinates on the fuzzy-sphere can be represented globally by three $N \times N$ matrices $\mathbf{X}_i$, $i = 1, 2, 3$, satisfying

$$\mathbf{X}_i \mathbf{X}_i = R_S^2 \mathbf{1}\,, \tag{2.6}$$

where $\mathbf{1}$ is the $N \times N$ unit matrix, with $\mathbf{X}_i$ proportional to the generators $\mathbf{L}_i$ of SU(2) in the irreducible $N \times N$ representation,

$$[\mathbf{L}_i, \mathbf{L}_j] = i\epsilon_{ijk}\mathbf{L}_k\,, \qquad \mathbf{L}_i\mathbf{L}_i = k(k+1)\mathbf{1}\,. \tag{2.7}$$

From this we deduce that

$$\mathbf{X}_i = \lambda_k \mathbf{L}_i \quad \Rightarrow \quad [\mathbf{X}_i, \mathbf{X}_j] = i\lambda_k \epsilon_{ijk}\mathbf{X}_k\,, \tag{2.8}$$

with

$$\lambda_k := \sqrt{\bar{a}}\, l_P \sqrt{\frac{2k+1}{k(k+1)}}\,. \tag{2.9}$$

In the large-N limit the $\mathbf{X}_i$ in equation (2.8) become commutative and the commutative sphere is recovered. However there is still a vestige of non-commutativity, even in the limit $k \to \infty$, since it is known, [18, 19], that the $k \to \infty$ limit of (2.8) describes a non-commutative plane under stereographic projection. This is seen by defining $\mathbf{X}_\pm = \mathbf{X}_1 \pm i\mathbf{X}_2$ and performing the analogue of stereographic projection for fuzzy co-ordinates:

$$\mathbf{Z} = \mathbf{X}_- \left(\mathbf{1} - \frac{\mathbf{X}_3}{R_S}\right)^{-1}, \qquad \mathbf{Z}^\dagger = \left(\mathbf{1} - \frac{\mathbf{X}_3}{R_S}\right)^{-1} \mathbf{X}_+\,. \tag{2.10}$$

Then, for large k,

$$[\mathbf{Z}, \mathbf{Z}^\dagger] = 2\lambda_k R_S \left(\mathbf{1} - \frac{\mathbf{X}_3}{R_S}\right)^{-2} + o\left(\frac{1}{k}\right). \tag{2.11}$$

Now, although the operator $\mathbf{X}_3/R_S$ has eigenvalues between -1 to $+1$ inclusive, only a very small range above -1 is necessary to cover the whole $\mathbf{Z}$-plane. To see this observe that

$$\frac{1}{2}(\mathbf{Z}\mathbf{Z}^\dagger + \mathbf{Z}^\dagger\mathbf{Z}) = R_S^2 \left(\mathbf{1} + \frac{\mathbf{X}_3}{R_S}\right)\left(\mathbf{1} - \frac{\mathbf{X}_3}{R_S}\right)^{-1} + o\left(\frac{1}{k}\right). \tag{2.12}$$

Writing $\mathbf{X}_3/R_S = -\mathbf{1} + \mathbf{T}/R_S^2$, where $\mathbf{T}/R_S^2$ has eigenvalues between 0 and 2, this reads

$$\frac{1}{2}(\mathbf{Z}\mathbf{Z}^\dagger + \mathbf{Z}^\dagger\mathbf{Z}) = \frac{1}{2}\mathbf{T}\left(\mathbf{1} - \frac{\mathbf{T}}{2R_S^2}\right)^{-1} + o\left(\frac{1}{k}\right). \tag{2.13}$$

Now, in the $k \to \infty$ limit, we can cover the whole of the $\mathbf{Z}$-plane by projecting all operators onto the subspace spanned by eigenvectors of $\mathbf{T}$ with eigenvalues in the range 0 to $\bar{a}\sqrt{k}\, l_P^2$. Hence, for $k \to \infty$, $\mathbf{T}/R_S^2 \to 0$ in (2.13) and we can replace $\mathbf{X}_3/R_S$ with $-\mathbf{1}$ in (2.11) to give

$$\mathbf{Z}\mathbf{Z}^\dagger + \mathbf{Z}^\dagger\mathbf{Z} = \mathbf{T} \qquad \text{and} \qquad [\mathbf{Z}, \mathbf{Z}^\dagger] = \theta \tag{2.14}$$

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with non-commutativity parameter

$$\theta = \lim_{k\to\infty} \frac{\lambda_k R_S}{2} = \lim_{k\to\infty} \frac{\bar{a} l_P^2}{2} \frac{(2k+1)}{\sqrt{k(k+1)}} = \bar{a} l_P^2 \,. \tag{2.15}$$

Thus, even though the event horizon becomes an ordinary commutative sphere as $k \to \infty$, there is still a vestige of the quantum theory left in that the neighbourhood of a point on the event horizon can be stereographically projected onto the non-commutative plane.

3. Rotating black holes

Now consider a rotating black hole with angular momentum $J^2 = j(j+1)\hbar^2$ and zero charge. The event horizon is still topologically a sphere, though not metrically a round sphere it still has a fuzzy description. The classical formula for the mass as a function of angular momentum and area (the Christodoulou-Ruffini mass [20]) is[1]

$$M^2 = \frac{1}{4}\bar{A} + \frac{J^2}{\bar{A}} \,, \tag{3.1}$$

or

$$\frac{\bar{A}}{2} = M^2 + \sqrt{M^4 - J^2} \tag{3.2}$$

(the positive square root is taken here because A is the area of the outer horizon). From the above formula comes the bound

$$J^2 \leq M^4 \,, \tag{3.3}$$

otherwise $\bar{A}$ becomes complex. Using (3.1) and (3.2) this is equivalent to

$$J^2 \leq \frac{1}{4}\bar{A}^2 \,. \tag{3.4}$$

Classically the maximum allowed angular momentum is when (3.3) is saturated:

$$J^2_{\max} = M^4 = \frac{1}{4}\bar{A}^2 \,. \tag{3.5}$$

Consider the quantum version of (3.5). Using $J^2_{\max} = j_{\max}(j_{\max}+1)\hbar^2$, together with the ansatz (2.1), gives

$$\left(j_{\max} + \frac{1}{2}\right)^2 = \bar{a}^2 \left(k + \frac{1}{2}\right)^2 + \frac{1}{4} \,. \tag{3.6}$$

Quantum mechanically the bound might not be saturated so all we can safely say is that

$$\left(j_{\max} + \frac{1}{2}\right)^2 \leq \bar{a}^2 \left(k + \frac{1}{2}\right)^2 + \frac{1}{4} \,. \tag{3.7}$$

[1]Here we use units in which $G_N = c^2 = 1$ to keep the formula clean, but $\hbar$ will be retained so as to highlight quantum phenomena. Hence $l_P^2 = {m_P}^2 = \hbar$.

Suppose that the bound is saturated in the limit of large k, and hence large $j_{\max}$, so that

$$\lim_{k\to\infty} \frac{J^2_{\max}}{M^4} = 1 \quad \Leftrightarrow \quad \lim_{k\to\infty} \frac{4J^2_{\max}}{\bar{A}^2} = 1 \quad \Leftrightarrow \quad \lim_{k\to\infty} \frac{j^2_{\max}}{k^2} = \bar{a}^2 \,. \tag{3.8}$$

Now the fuzzy sphere is associated with a Hilbert space whose maximum angular momentum is k, so it seems very natural to take $j_{\max} = k$, in which case $\bar{a} = 1$. Then (3.5) must be modified to read

$$\frac{J^2_{\max}}{\hbar^2} = j_{\max}(j_{\max}+1) = \frac{1}{4}\left(\frac{\bar{A}^2}{\hbar^2} - 1\right) \tag{3.9}$$

with

$$\bar{A} = (2k+1)\hbar \,. \tag{3.10}$$

Note that a $k = 0$ black hole necessarily has $j = 0$ and is therefore a boson with spin zero.

It is possible that there is a correlation between k and j, even away from extremality, and that integral j implies integral k and half-integral j implies half-integral k. Indeed the area spectrum found in [11] for non-rotating black holes requires integral k when $j = 0$ for a hole carrying zero charge, half-integral k only appear for charged black holes in their analysis. The spectrum found in [12] for zero charge requires that j and k are both integral. The fuzzy sphere approach here does not impose any such restrictions. While a correlation between integral k and j, requiring that they be either both integral or both half-integral, seems plausible we have not found a proof that it is necessary.

The fact that the difference between the quantum bound (3.9) and the classical bound (3.5) is independent of A is a direct consequence of the choice $\bar{a} = 1$.

Equation (3.1) now reads

$$M^2 = \left\{ \frac{k(k+1) + j(j+1) + \frac{1}{4}}{(2k+1)} \right\} \hbar \,. \tag{3.11}$$

The mass of a black hole of a given area (fixed k) with maximum allowed angular momentum is now

$$M^2(J_{\max}) = \frac{1}{4}\left\{ \frac{8k(k+1)+1}{2k+1} \right\} \hbar \,. \tag{3.12}$$

In the quantum theory equation (3.5) is then replaced with

$$J^2_{\max} = M^4(J_{\max}) - \frac{\hbar^4}{16\bar{A}^2} = \frac{1}{4}(\bar{A}^2 - \hbar^2) \,, \tag{3.13}$$

so (3.3) is never saturated for finite k. In terms of j and k the bound is

$$(2k+1)^2 > 4j(j+1) \,. \tag{3.14}$$

4. Charged black holes

Including electric charge Q_e the classical Christodoulou-Ruffini formula reads

$$M^2 = \frac{1}{\bar{A}}\left\{ \frac{1}{4}(\bar{A} + Q_e^2)^2 + J^2 \right\} \tag{4.1}$$

or, if magnetic monopoles with charge Q_m are also included,

$$M^2 = \frac{1}{\bar{A}}\left\{\frac{1}{4}(\bar{A}+Q^2)^2 + J^2\right\} \tag{4.2}$$

where

$$Q^2 = Q_e^2 + Q_m^2\,. \tag{4.3}$$

With $\bar{A} = (2k+1)\hbar$ and Q_e quantised in multiples of the electric charge e the quantum version of (4.2) becomes

$$M^2 = \left\{\frac{\left[2k+1+\alpha q_e^2 + \alpha^{-1}(q_m/2)^2\right]^2 + 4j(j+1)}{4(2k+1)}\right\}\hbar\,, \tag{4.4}$$

with q_e and q_m integers (we use units with $4\pi\epsilon_0 = 1$ so that the fine structure constant is $\alpha = e^2/\hbar$ when $c = 1$, the factor of $\alpha^{-1}/4$ multiplying q_m^2 allows for the Dirac quantisation condition, $Q_e Q_m = \widetilde{N}\hbar/2$ where $\widetilde{N}$ is an integer). Thus, as suggested in [5], the black hole mass is characterised by four discrete numbers: k and j, which can each be either integral or half-integral, and q_e and q_m which are both integers.

This particle picture of black holes has also been a central theme in the work of 't Hooft, [6, 8]. The general form of the spectrum (4.4) was derived by Bekenstein [5], the new ingredient here is that some of the constants differ as a consequence of the hypothesis that the event horizon is modeled by a fuzzy sphere.

Demanding that $\bar{A}$ in (4.2) is real gives the classical bound

$$M^4 - Q^2M^2 - J^2 \geq 0\,. \tag{4.5}$$

Defining

$$\Delta^2 := M^4 - Q^2M^2 - J^2 \tag{4.6}$$

(4.2) can be used to express Δ^2 in terms of the area

$$\Delta^2 = \frac{(\bar{A}^2 - Q^4 - 4J^2)^2}{16\bar{A}^2}\,. \tag{4.7}$$

Actually for the outer horizon

$$\bar{A}^2 \geq Q^4 + 4J^2\,, \tag{4.8}$$

so we can conclude that

$$\Delta = \frac{\bar{A}^2 - Q^4 - 4J^2}{4\bar{A}} \geq 0\,. \tag{4.9}$$

The classical bound (4.5) is thus saturated when

$$\bar{A}^2 = Q^4 + 4J^2\,. \tag{4.10}$$

Quantum mechanically this bound cannot always be achieved, the best we can hope to do is minimise Δ. The quantum version of Δ is, using (3.10),

$$\Delta/\hbar = \frac{(2k+1)^2 - \left(\alpha q_e^2 + \alpha^{-1}(q_m/2)^2\right)^2 - 4j(j+1)}{4(2k+1)}\,. \tag{4.11}$$

The special case $q_e = q_m = 0$ reproduces the analysis in the previous section.

The final conclusion here is that, for q_e, q_m and j given, k (or equivalently the area) is bounded below by

$$(2k+1)^2 \geq \left(\alpha q_e^2 + \alpha^{-1}\left(\frac{q_m}{2}\right)^2\right)^2 + 4j(j+1)\,. \tag{4.12}$$

5. Entropy

Using Hawking's result for the entropy, at least for large mass black holes, we have

$$S = \frac{A}{4l_P^2} = \pi\frac{\bar{A}}{l_P^2} = (2k+1)\pi\,, \tag{5.1}$$

in units with $k_B = 1$.

In the fuzzy sphere picture presented here it seems natural to guess that a $k = 0$ black hole should have zero entropy, since it does not appear to have any internal degrees of freedom. The simplest modification of (5.1) compatible with this hypothesis is to take

$$S = 2k\pi = \frac{A}{4l_P^2} - \pi\,, \tag{5.2}$$

but then the number of microstates would not be an integer in general.

In any case we expect the number of microstates of the black hole for large k to be

$$\Omega \approx \exp(2k\pi)\,, \tag{5.3}$$

using $S = \ln\Omega$. Without a more detailed understanding of the microscopic states however, this formula cannot be verified directly.

String theory provides a way of calculating the entropy of a black hole directly from the number of microscopic states. The first calculations, [21], were done for 5-dimensional black holes, with 3-dimensional event horizons S^3 which are not symplectic manifolds (fuzzy descriptions of S^3 do exist though, [22]). A string theory derivation of the entropy of extremal supersymmetric black holes in 4-dimensions was given in [23] and generalised to the non-extremal case in [24]. The entropy calculated in [23] agreed with the earlier explicit evaluations of the black hole area [25] and reproduced Hawking's factor 1/4. The upshot of the analysis in [23] is that the entropy depends on four integers, labelled Q_2, Q_6, n and m in their notation, and, for large integers, is given by

$$S = 2\pi\sqrt{Q_1 Q_6 n m}\,. \tag{5.4}$$

Clearly this agrees with the result (5.1) if, at least for large k,

$$k^2 \approx Q_1 Q_6 n m\,. \tag{5.5}$$

In general one expects corrections to this formula of order k.

It is not obvious how the string theory arguments might relate to (2.1). The problem is that the string theory calculation must be carried out in the regime of small string coupling

g_s, where the notion of a black hole is not well defined. Black holes, it is believed, emerge from string theory as classical objects only for large g_s and one relies on supersymmetry to argue that the small g_s calculation still gives the correct answer for the entropy even when g_s is large. But there is no analogue of (2.1) in string theory when g_s is small. It has been suggested that fuzzy spheres can be viewed as spherical D2-branes [26] and they also emerge as ground states of matrix models [27], so it may prove possible to investigate the ideas presented here directly in string theory.

Attempts have also been made to calculate the entropy of black holes in the loop approach to quantum gravity (see [28] and references therein). An equal spaced area spectrum like that of (1.2) was found in [13] though the prefactor was undetermined, and the question of sub-leading corrections has also been addressed [29]. For a discussion of entropy from a fuzzy sphere approach within the loop quantum gravity framework see [30].

6. Conclusions

By modeling the event horizon of a black hole as a fuzzy sphere, and identifying the maximum angular momentum of the black hole with the maximum angular momentum associated with the Hilbert space underlying the fuzzy sphere, the mass spectrum of a quantum black hole has been suggested. The spectrum is

$$M^2_{k,j,q_e,q_m} = \left\{ \frac{\left[2k+1+\alpha q_e^2 + \alpha^{-1}(q_m/2)^2\right]^2 + 4j(j+1)}{4(2k+1)} \right\} {m_P}^2 , \tag{6.1}$$

where k is either an integral or a half-integral quantum number determining the area of the event horizon,

$$A = 4\pi(2k+1)l_P^2 , \tag{6.2}$$

j is the angular momentum quantum number, q_e the electric charge and q_m the monopole charge. For given values of j, q_e and q_m the quantum number k is bounded below by the quantum analogue of the familiar classical bound,

$$(2k+1)^2 \geq 4j(j+1) + \left(\alpha q_e^2 + \alpha^{-1}\left(\frac{q_m}{2}\right)^2\right)^2 . \tag{6.3}$$

In general the quantum bound (6.3) cannot be saturated unless α takes on special values, for example when $j = q_m = 0$ the bound can be saturated if α is rational. The area quantisation (6.2) has also been found in a semi-classical approach [10] and in a mini-superspace approach to quantising black holes [11, 12], but in [11] it is argued that α must be rational, whereas the fuzzy sphere approach presented here does not seem to require this.

Equation (6.1) is a version of a suggestion of Bekenstein's, but with different constants. The constants have been fixed here by assuming that the maximum angular momentum of a rotating hole be identified with the maximum angular momentum of the underlying Hilbert space of the fuzzy sphere, thus side-stepping the question of the microscopic degrees of freedom. If this assumption is relaxed, then the above formulae still apply but with the undetermined parameter $\bar{a}$ re-introduced so that $2k+1$ is replaced with $(2k+1)\bar{a}$ everywhere. Bekenstein used $\bar{a} = (\ln 2)/\pi$ in [5] and Hod found $\bar{a} = (\ln 3)/\pi$ in [31]. Bekenstein's value of $\ln 2/\pi$ is a consequence of associating classical bits with each unit of phase space area and calculating the number of possible configurations. In a fully quantum approach it would seem more natural to use qubits rather than classical bits, as suggested in [30], and to determine the entropy from a density matrix, but any explicit expression would require making further, more specific, assumptions about the allowed quantum states. Alternatively it may be possible to determine the entropy using a specific field theory on the fuzzy sphere, perhaps a supersymmetric field theory, or by viewing the fuzzy sphere as a dynamical object in the context of a matrix model. From this last point of view it is intriguing that studies of the ground state fuzzy sphere in matrix Chern-Simons theory [32] reveal a radius that scales as $\sqrt{N}$, which is precisely what is needed for the black-hole interpretation advanced here.

Assuming that $\bar{a} = 1$ we see that the smallest mass (the ground state) given by this formula is

$$M = \frac{1}{2} m_P = 6.10 \times 10^{18}\,\mathrm{GeV}/c^2\,, \tag{6.4}$$

when $k = j = q_e = q_m = 0$. The event horizon area for the minimum mass black hole is

$$A = 4\pi l_P^2\,. \tag{6.5}$$

The next smallest mass is for a non-rotating black hole carrying a single unit of charge $q_e = 1$ with $k = q_m = 0$, which lies

$$\Delta M = \frac{1}{2}\alpha m_P \tag{6.6}$$

above the ground state. The numerical value here depends on the value of α used. One should take into the account running of the coupling constant and use U(1) hypercharge rather than electric charge, or some other U(1) charge depending on new physics.

As a by-product of the analysis we have seen that, as $k \to \infty$, the neighbourhood of a point on the event horizon can be projected stereographically onto a non-commutative plane with non-commutativity parameter

$$\theta = l_P^2 \tag{6.7}$$

(if $\bar{a} \neq 1$ this equation is replaced with $\theta = \bar{a} l_P^2$).

Our analysis has avoided any discussion of the microscopic degrees of freedom of the black hole. In particular little has been said about entropy beyond using Hawking's formula to determine the entropy from the area. This formula may well be modified for small black holes by quantum phenomena, but without a more detailed understanding of the black hole microstates it is impossible to be more specific at this stage.

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Erratum

It was states erroneously in the abstract and on page 4 that the event horizon becomes a commutative sphere in the classical limit, but this conclusion is incorrect. Locally the event horizon becomes a non-commutative plane in the limit of large black-holes.

Even though $\lambda_k = \sqrt{\frac{(2k+1)\bar{a}}{k(k+1)}}\, l_P$ vanishes as $N = 2k+1 \to \infty$ this does not mean that the $\mathbf{X}_i$ in

$$[\mathbf{X}_1, \mathbf{X}_2] = i\lambda_k \mathbf{X}_3 = i\lambda_k^2 \mathbf{L}_3$$

become commutative in this limit.

Heuristically this can be seen by focusing on a region near the south pole of a large black-hole, in the limit of large k. At the south pole $\mathbf{X}_1$ and $\mathbf{X}_2$ are transverse to the surface and $\mathbf{X}_3$ is normal to it, with $\mathbf{X}_3 \approx -R_s$.

In a basis in which

$$\mathbf{L}_3 = \begin{pmatrix} k & & \\ & \ddots & \\ & & -k \end{pmatrix}$$

is diagonal the eigenvalue $\mathbf{X}_3 \approx -R_s$ corresponds to the minimum eigenvalue $-k$ of $\mathbf{L}_3$ so

$$[\mathbf{X}_1, \mathbf{X}_2] \approx -i\lambda_k^2 k$$

and, as $k \to \infty$,

$$[\mathbf{X}_1, \mathbf{X}_2] = -2i\bar{a} l_P^2.$$

Hence, in an infinitesimal region around the pole, the event horizon looks like a non-commutative plane in the infinite k limit, a conclusion that is supported by the stereographic projection analysis in the original paper.

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